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Dislocation theory for geophysical applications

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A fault plane which has undergone slip over a limited area, a thin intrusion or a crack whose faces have been caused to slide over one another or separate by the action of an applied stress are all physical realizations of a dislocation, that is, an internal surface in an elastic solid across which there is a discontinuity of displacement. Since this discontinuity varies from point to point of the internal surface it is actually a so-called Somigliana dislocation. It can, however, be built up from the more familiar dislocations of crystal physics which have a constant displacement discontinuity.

Methods of finding the elastic displacement field around a dislocation in a solid with free surfaces will be outlined. The field of an infinitesimal dislocated area in a semi-infinite solid can be found quite simply, and from it the field of a general dislocation can be obtained by integration. The energy associated with a dislocation is discussed in connexion with energy release in earthquakes.

1. INTRODUCTION: SOMIGLIANA DISLOCATIONS

Figure 1*a* represents a fault plane in which relative slip of the faces is confined to the interior of the curve *C*. Figure 1*b* shows a cross-section of, say, a thin igneous intrusion. Figure 1*c* represents the collapse of a worked-out coal-seam. These are all examples of so-called Somigliana dislocations (Somigliana 1914, 1915; Gebbia 1902). The formal construction of a Somigliana dislocation in an

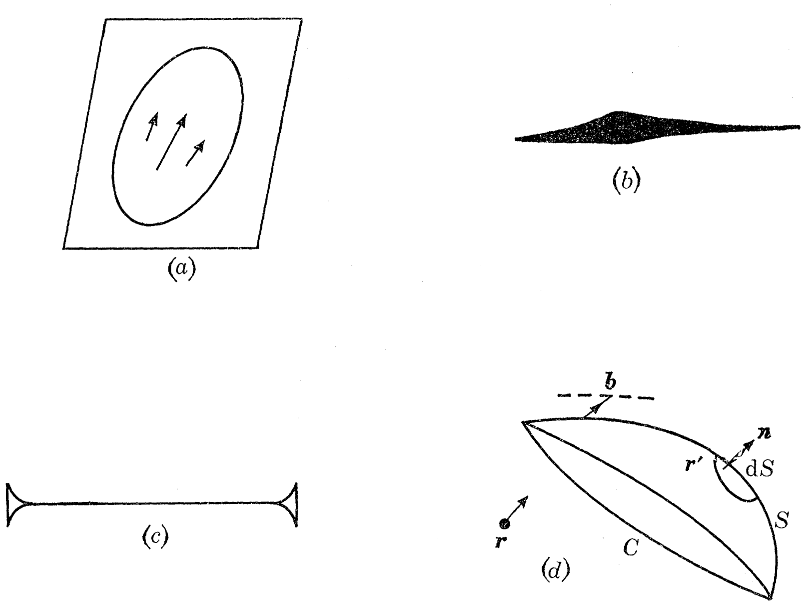

FIGURE 1. Somigliana dislocations.

elastic continuum goes as follows: make a cut over a surface *S* (figure 1*d*) (not necessarily plane) bounded by a curve *C* and give the two faces of the cut a relative displacement $\boldsymbol{b}(\boldsymbol{r})$ which varies from point to point $\boldsymbol{r}$ of *S*. Where this leaves a gap (as in figure 1*b*) fill it with a layer of the same material, and where there would be interpenetration of matter (as in figure 1*c*) prevent this by scraping away a layer on one face or the other. Finally weld the material together again.

Cracks can be treated in terms of Somigliana dislocations. Suppose that a crack forms in a body under an applied stress, and for simplicity let the faces of the crack slide over one another but not separate. We may say that the applied stress has induced in the medium a Somigliana dislocation of the kind shown in figure 1*a*. If the faces of the crack are freely slipping the discontinuity vector $\boldsymbol{b}$ will adjust itself in such a way that the surface tractions on S due to the applied stress and to the dislocation exactly cancel. If there is friction between the faces the cancellation may be imperfect. In the freely slipping case the dislocation disappears when the applied stress is removed. With friction the surfaces may stick at some stage of the unloading to leave behind a 'fossilized' remnant of the Somigliana dislocation originally produced by the applied stress.

A crack whose faces separate under load may be looked at in the same way; as long as it is held open the elastic field is unaffected if we fail to fill in the gap according to the recipe above.

If the displacement discontinuity $\boldsymbol{b}$ is known as a function of position over S the elastic field (displacement, strain, stress) which it produces, and the energy associated with it, can be calculated, fairly easily if it is remote from free surfaces of the medium and otherwise with more or less complicated corrections for the presence of the free surfaces (§2).

Somigliana dislocations are well suited to describe geophysical discontinuities, but the bulk of modern dislocation theory (Nabarro 1967) considers only the subclass of them, Volterra dislocations, for which $\boldsymbol{b}$ is constant over the discontinuity surface. In §3 we describe some of their properties and indicate how, if necessary, they can be used to synthesize Somigliana dislocations.

2. The elastic field of a Somigliana dislocation

We shall use the following sign convention relating the discontinuity $\boldsymbol{b}$ and the normal $\boldsymbol{n}$ to S (figure 1*d*). Draw an arrow (with head and tail) piercing S and defining the direction of its normal. Then $\boldsymbol{b}$ is the displacement on the head side of S minus the value on the tail side.

To find the elastic displacement produced by a Somigliana dislocation in an infinite medium imagine that, in addition to the dislocation, there is also a concentrated point force acting at the point $\boldsymbol{r}$ at which we wish to find the displacement $u_i(\boldsymbol{r})$ due to the dislocation (figure 1*d*).

If we introduce the dislocation and then apply the force the work done on the medium is simply the sum of the amounts E_{D} and E_{F} of work done when either the dislocation or the force is introduced by itself, because, in the linear theory of elasticity, the response of a body to external loading is unaffected by internal stresses. On the other hand, if first the force and then the dislocation is introduced the work done is $E_{\mathrm{F}}+E_{\mathrm{D}}$ plus two extra terms. One of these is the work done by the point force when its point of application shifts by the displacement which the dislocation produces at $\boldsymbol{r}$. The other is the work done by the surface tractions (produced by the point force) on the two faces of the cut as the faces separate by $\boldsymbol{b}$. But since the final state of the medium is the same whether the dislocation or the point force is introduced first, the sum of these two extra terms must be zero. This leads at once to the relation

$$u_i(\boldsymbol{r}) = \int_S b_j(\boldsymbol{r}')\, p_{jk}^{(i)}(\boldsymbol{r}')\, n_k \,\mathrm{d}S, \tag{2.1}$$

where $p_{jk}^{(i)}(\boldsymbol{r}')$ is the stress produced at a point $\boldsymbol{r}'$ on S by a point force of unit magnitude at $\boldsymbol{r}$ parallel to the x_i axis, and n_k is the normal to S at $\boldsymbol{r}'$. The left-hand side is the work done by the unit force moving through the dislocation displacement and the right-hand side is minus the work done by the surface tractions at the cut. (For the sign of the second term compare the discussion following equation (2.5) below.)

In an isotropic medium with Lamé constants λ and μ and the Poisson ratio σ the Hooke law relation between $p_{jk}^{(i)}$ and the corresponding displacement can be written in the form

$$p_{jk}^{(i)} = \lambda \frac{\partial U_{il}}{\partial x_l'} \delta_{jk} + \mu \frac{\partial U_{ij}}{\partial x_k'} + \mu \frac{\partial U_{ik}}{\partial x_j'} \tag{2.2}$$

where U_{il} is the x_i-component of the displacement at $\boldsymbol{r}(x_1, x_2, x_3)$ due to a unit force acting at $\boldsymbol{r}'(x_1', x_2', x_3')$ parallel to the x_l-axis. If we use the explicit form

$$U_{il} = \frac{1}{8\pi\mu}\left[\delta_{il}\nabla^2 - \frac{1}{2(1-\sigma)}\frac{\partial^2}{\partial x_i\,\partial x_l}\right]|\boldsymbol{r}-\boldsymbol{r}'| \tag{2.3}$$

(Love 1927), and note that $\partial/\partial x_m'$ is equivalent to $-\partial/\partial x_m$ when acting on $|\boldsymbol{r}-\boldsymbol{r}'|$, equation (2.1) becomes

$$u_i(\boldsymbol{r}) = \frac{1}{8\pi(1-\sigma)}\vartheta_{ijk} I_{jk},$$

with (Eshelby 1961)

$$\vartheta_{ijk} = \frac{\partial^3}{\partial x_i\,\partial x_j\,\partial x_k} - \left\{\sigma\delta_{jk}\frac{\partial}{\partial x_i} + (1-\sigma)\,\delta_{ij}\frac{\partial}{\partial x_k} + (1-\sigma)\,\delta_{ik}\frac{\partial}{\partial x_j}\right\}\nabla^2$$

and
$$I_{jk} = \int_S b_j(\boldsymbol{r}')\,|\boldsymbol{r}-\boldsymbol{r}'|\,n_k\,\mathrm{d}S.$$

If the discontinuity surface is flat and happens to lie in a plane x_3 = const. I_{j1} and I_{j2} are zero and one only needs to calculate the three surface integrals

$$I_{j3} = \int_S b_j(\boldsymbol{r}')\,|\boldsymbol{r}-\boldsymbol{r}'|\,\mathrm{d}x_1'\,\mathrm{d}x_2'.$$

The rest of the calculation is mere differentiation. We may regard (2.1) as exhibiting a finite Somigliana dislocation as a mosaic of infinitesimal dislocations of area dS, normal n_k and a discontinuity vector b_j which is, of course, effectively constant all over the small area dS. The displacement produced by one of these elementary dislocations is

$$\mathrm{d}u_i = b_j n_k \,\mathrm{d}S\, p_{jk}^{(i)}, \tag{2.4}$$

with $p_{jk}^{(i)}$ in the form (2.2). According to (2.3) we have $U_{il}(\boldsymbol{r}, \boldsymbol{r}') = U_{il}(\boldsymbol{r}', \boldsymbol{r})$ and so (2.4) with (2.2) can be given a new interpretation: du_i can be derived from the displacement U_{il} at the point of observation $\boldsymbol{r}$ due to a point force acting not, as hitherto, at $\boldsymbol{r}$, but rather at $\boldsymbol{r}'$. To do this we have, according to (2.2) only to differentiate U_{il} with respect to the x_m', the coordinates of the point of application, and form a suitable linear combination of the derivatives. In other words, du_i is the same as the displacement produced at $\boldsymbol{r}$ by a collection of force-doublets situated at the position $\boldsymbol{r}'$ of the elementary dislocation. If the plane of the dislocation is a plane x_2 = const. and the displacement discontinuity is $(b, 0, 0)$, parallel to the x_1-axis, figure 2*a*, or, equally, the plane is parallel to x_1 = const., and the discontinuity is $(0, b, 0)$, figure 2*b*, we have

$$\mathrm{d}u_i = \mu b\,\mathrm{d}S\left(\frac{\partial U_{i1}}{\partial x_2'} + \frac{\partial U_{i2}}{\partial x_1'}\right),$$

which says that du_i can be produced by a pair of equal and opposite force couples each of moment $\mu b\,\mathrm{d}S$ force–times–distance units, figure 2*c*. If, on the other hand, the discontinuity vector is $(0, b, 0)$ and so normal to the plane x_2 = const. of the dislocation (figure 2*e*) we have

$$\mathrm{d}u_i = b\,\mathrm{d}S\left[\lambda\frac{\partial U_{il}}{\partial x_1'} + \lambda\frac{\partial U_{i3}}{\partial x_3'} + (\lambda+2\mu)\frac{\partial U_{i2}}{\partial x_2'}\right],$$

which is the displacement due to three 'double forces without moment' (Love 1927) which despite their name have, in an obvious sense, a moment (not a couple) which is actually $(\lambda+2\mu)\,b\,\mathrm{d}S$ for the vertical doublet and $\lambda b\,\mathrm{d}S$ for the two horizontal ones (figure 2*f*). If we had chosen to discuss the elementary dislocations of figure 2*a*, *b* in a coordinate system rotated through 45° in the plane of the paper we should have got the forces shown in figure 2*d*, made up of two double forces of moment $\pm\mu b\,\mathrm{d}S$ which in fact produce the same displacement as the couples in figure 2*c*.

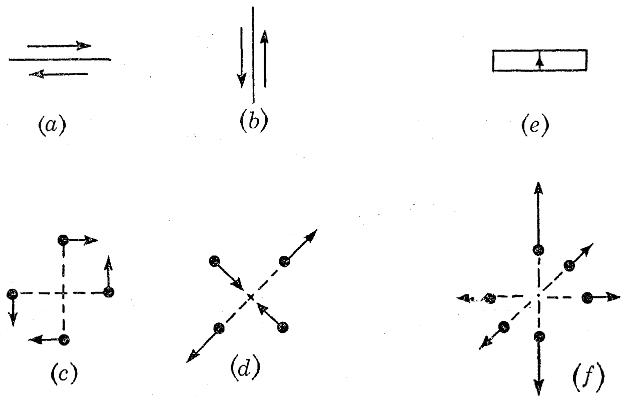

FIGURE 2. Equivalence between elementary dislocated areas and point-force clusters.

The displacement due to a dislocation in a body with stress-free surface Σ, say, can be found from the displacement u_i of the same dislocation in an infinite medium in the following way. Suppose that u_i produces a surface traction $p_{ij}n_j$ on Σ. Calculate the displacement u_i' due to surface tractions $-p_{ij}n_j$ applied to the surface of a finite body bounded by Σ. Then u_i+u_i' is the required displacement since it has the required singularity inside Σ and gives zero traction on Σ.

For the case of a semi-infinite solid bounded by a stress-free plane surface (which should cover many geophysical applications) a simple alternative treatment is possible. We go back to (2.4) which says that the elastic field of an elementary dislocation in an infinite medium is the same as that of a certain collection of point forces operating in an infinite medium. To adapt this expression to give the displacement in a half-space with, say, the surface $x_3 = 0$ stress-free it is evidently only necessary to take U_{il} in (2.4) to stand, as before, for the x_i- component of displacement at $\boldsymbol{r}$ due to a point force acting parallel to the x_l-axis at $\boldsymbol{r}'$, but now in a semi-infinite medium with the plane $x_3 = 0$ stress-free. Formation of doublets by differentiating this new U_{il} with respect to x_m' will not upset the fact that $x_3 = 0$ is stress-free. Integration will then give the field of a finite dislocation in the semi-infinite medium.

The necessary results for the displacement due to a point force in a semi-infinite medium with a stress-free surface have been given with various degrees of explicitness by Mindlin (1936), Westergaard (1952), Lur'e (1964) and Solomon (1968). Mindlin & Cheng (1950) give the field of doublets of the kind shown in figure 2. The result of inserting one of these expressions into (2.2) plus (2.4) is decidedly complicated. Steketee (1958) has written out the displacement for an elementary dislocation like the one in figure 2*a* with the plane of relative slip parallel to the free surface for the special case where the Poisson ratio is $\frac{1}{4}$ $(\lambda = \mu)$.† Even then the result is rather cumbersome (see also Maruyama 1964).

The energy change associated with the appearance or disappearance of a discontinuity surface

† In Steketee's equations (7.18) and (7.20) τ is a misprint for w.

is an important quantity. We begin with the case of the formation of a Somigliana dislocation in an initially stress-free region.

The elastic energy of a body is given by the integral

$$W = \frac{1}{2}\int u_i p_{ij} n_j \, \mathrm{d}S$$

taken over its surface. For a Somigliana dislocation the effective surface is made up of the two faces of the cut on either side of the discontinuity surface S (figure 1 *d*). If p^{D}_{ij} is the stress produced by the dislocation the surface traction $p^{\mathrm{D}}_{ij} n_j$ is continuous across S although the individual components of p^{D}_{ij} are not. Since the two faces of the cut are displaced relatively by $\boldsymbol{b}$ the energy required to establish the dislocation becomes

$$W = -\frac{1}{2}\int_S b_i p^{\mathrm{D}}_{ij} n_j \, \mathrm{d}S. \tag{2.5}$$

This is a positive quantity despite the negative sign which comes about as follows. In figure 1*d* the face of the cut on, say, the convex side of S has a normal, pointing out of the material, which is in the opposite direction to the normal we have assigned to S. Combined with the sign convention for $\boldsymbol{b}$ (§ 2) this gives the minus sign in (2.5). The quantity (2.5) is also, of course, the energy released if the dislocation disappears.

If the dislocation forms in the presence of a pre-existing stress p^{A}_{ij} we have to take account of the work done at the discontinuity surface against the traction $p^{\mathrm{A}}_{ij} n_j$ and the work done is

$$W = -\int_S (p^{\mathrm{A}}_{ij} + \tfrac{1}{2} p^{\mathrm{D}}_{ij})\, b_i n_j \, \mathrm{d}S, \tag{2.6}$$

which can also be written more symmetrically as

$$W = -\int_S \tfrac{1}{2}(p^{\mathrm{I}}_{ij} + p^{\mathrm{F}}_{ij})\, b_i n_j \, \mathrm{d}S, \tag{2.7}$$

where $p^{\mathrm{I}}_{ij} = p^{\mathrm{A}}_{ij}$ and $p^{\mathrm{F}}_{ij} = p^{\mathrm{A}}_{ij} + p^{\mathrm{D}}_{ij}$ are the initial and final stresses. As explained in § 1, when a freely slipping crack forms a Somigliana dislocation is generated which completely annuls the traction $p^{\mathrm{A}}_{ij} n_{ij}$ over S, so that $p^{\mathrm{F}}_{ij} = 0$, and (2.6) or (2.7) gives a negative energy of formation

$$W = -\frac{1}{2}\int_S p^{\mathrm{A}}_{ij} b_i n_j \, \mathrm{d}S = +\frac{1}{2}\int_S p^{\mathrm{D}}_{ij} b_i n_j \, \mathrm{d}S,$$

so that, comparing with (2.5), there is an energy release when the crack appears which is the same as the energy released when an equivalent dislocation disappears in the absence of an external stress.

Two remarks should perhaps be made about (2.6). The first is that although it represents the work required to form the dislocation in the presence of p^{A}_{ij}, not all this work goes into elastic energy; some of it goes to increase the potential energy of the loading mechanism responsible for p^{A}_{ij}, in geophysical situations ultimately the Earth's gravitational field. The other is that it is assumed that p^{A}_{ij} does not vary appreciably as the dislocation forms (or disappears), a condition which will often, but not always, be fulfilled.

The integrals (2.6) and (2.5) only give the total energy released when a discontinuity appears or disappears. If we actually know how the displacement discontinuity varies with time during these processes the radiation field can in principle be calculated starting from a generalization to time-dependent $\boldsymbol{b}$ of the expression (2.4) for the field of an elementary dislocation (Nabarro 1951; Eshelby 1962; Kosevich 1965).

3. Volterra dislocations

The modern theory of dislocations (Nabarro 1967) confines itself almost entirely to the special case where the displacement discontinuity $\boldsymbol{b}$ is a constant. We shall call such dislocations Volterra dislocations, though strictly speaking they are a combination of Volterra's dislocations of the first, second and third kinds only.

If $\boldsymbol{b}$ is constant (2.1) can be transformed into

$$\boldsymbol{u}(\boldsymbol{r}) = -\frac{\boldsymbol{b}}{4\pi}\Omega - \frac{1}{4\pi}\int_C \frac{\boldsymbol{b}\times d\boldsymbol{l}}{R} + \frac{1}{4\pi}\frac{1}{2(1-\sigma)}\,\text{grad}\int_C \frac{\boldsymbol{b}\times d\boldsymbol{l}\cdot\boldsymbol{R}}{R} \tag{3.1}$$

(Burgers 1939). Here

$$\boldsymbol{R} = \boldsymbol{r}-\boldsymbol{r}', \quad R = |\boldsymbol{r}-\boldsymbol{r}'|,$$

$d\boldsymbol{l}$ is an element of the curve C at $\boldsymbol{r}'$ and

$$\Omega = \int_S \frac{\boldsymbol{R}\cdot\boldsymbol{n}}{R^3}\,dS$$

is the solid angle subtended by S at $\boldsymbol{r}$. (To verify that (3.1) agrees with (2.1) for constant $\boldsymbol{b}$ use Stokes's theorem to turn the line integrals into surface integrals.)

It is known that the gradient of Ω is independent of S and consequently also continuous across it. Hence the gradient of $\boldsymbol{u}$, and with it the strain, rotation, and stress around the Volterra dislocation, are independent of S. If this continuous gradient is now integrated to recover the displacement we find that $\boldsymbol{u}$ is a multiple-valued quantity which changes by $\pm\boldsymbol{b}$ each time the curve C is encircled by the path of integration. It is almost equivalent to say that $\boldsymbol{u}$ is only defined modulo $\boldsymbol{b}$, i.e. to within a multiple of $\boldsymbol{b}$. Because of this we cannot observe the discontinuity surface of a Volterra dislocation with discontinuity vector $\boldsymbol{b}$ in a crystal with lattice parameter $\boldsymbol{b}$, because (unless we actually watch the process of displacement) the displacement of one of a lattice of identical atoms can only be measured modulo $\boldsymbol{b}$. It is these facts which make Volterra dislocations so important in crystal physics. Such a dislocation can be regarded as a line singularity (dislocation loop) characterized by a curve C and a constant vector $\boldsymbol{b}$, known in this connection as the Burgers vector.

The simplest kind of Volterra dislocation is one where the curve C (the dislocation line) is an infinite straight line and the discontinuity surface S is a half-plane bonded by C. In figure 3*a* the point C indicates the trace of the curve C and CA is the trace of S. If the Burgers vector $\boldsymbol{b}$ is parallel to the curve C, and so perpendicular to the plane of the figure, we have a *screw* dislocation. The cross products $\boldsymbol{b}\times d\boldsymbol{l}$ in the line integrals of (3.1) are then zero, and the term in Ω gives a displacement

$$u_3 = \frac{b}{2\pi}\theta, \quad u_1 = 0, \quad u_2 = 0, \tag{3.2}$$

where the angle θ is as indicated in the figure and is limited by $-\pi \leqslant \theta \leqslant \pi$ so as to produce a discontinuity across CD rather than across some other line through C.

The plane AE can be made stress-free by introducing a screw dislocation of opposite sign at the 'image' point C′ so that the displacement due to a screw dislocation at C in a semi-infinite medium bounded by the stress-free plane BE is

$$u_3 = \frac{b}{2\pi}(\theta-\theta'). \tag{3.3}$$

The origin of θ' has been chosen so that the displacement is precisely $\pm\frac{1}{2}b$ on either side of D, and so that if θ', like θ, is limited by $-\pi \leqslant \theta' \leqslant \pi$ the discontinuity surface associated with θ' does not intersect the medium.

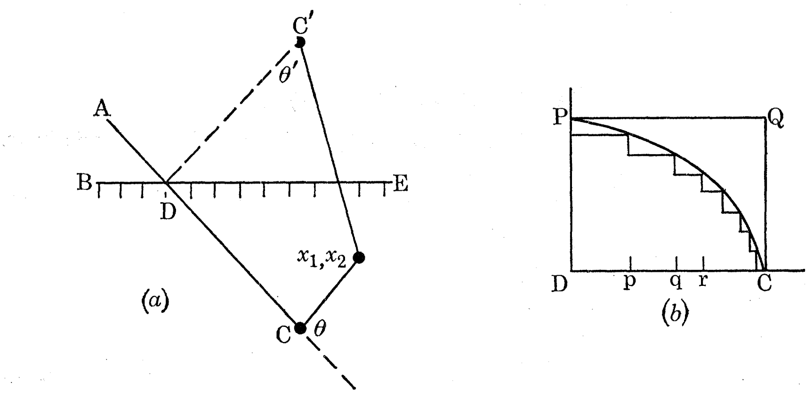

FIGURE 3. (*a*) Screw edge dislocations in a half-space. (*b*) Synthesis of a fault with variable slip from elementary screw and edge dislocations.

In geological terms the situation described by figure 3*a* and equation (3.3) represents a strike-slip fault, but one with the special property that the slip is constant from D to C and zero beyond C. How do we describe a strike-slip fault when the slip varies in some other way? We can, of course, go back to the general Somigliana dislocation. Alternatively, we can build up the fault using Volterra dislocations. Figure 3*b* shows the relative slip plotted as ordinate along DC. The line PQC represents the case we have been considering, constant slip produced by a single dislocation with Burgers vector b at C. Suppose that in fact the slip tapers off as shown by the smooth curve. Introduce n dislocations with equal Burgers vectors $b_0 = b/n$ at points p, q, r, ... Since the slip changes by b_0 each time a dislocation is passed the n dislocations will produce a stepped slip curve, and by choosing the positions of p, q, r, ... suitably the stepped curve may be made to agree with the smooth curve at n points. By increasing n (and thus decreasing b_0) the agreement can be improved indefinitely. In the limit $n \to \infty$, $b_0 \to 0$ we can say that the fault, actually a Somigliana dislocation, has been simulated by a continuous distribution of infinitesimal Voltarra dislocations. (For a general treatment on these lines see Bilby & Eshelby 1968; for a geophysical application see Weertman 1964.)

Suppose next that the relative slip of the two faces of the slip plane DC is in the plane of the figure instead of perpendicular to it, and of course parallel to DC. If the relative slip is constant along the slip plane as far as C and zero beyond it we have an *edge* dislocation. Its field is considerably more complex than (3.2) or (3.3).

In geological terms the edge dislocation is a dip-slip fault for which the relative slip is constant. The more realistic case of non-constant slip can be treated in the way already described for the strike-slip case, except that, in figure 3*b*, p, q, r, ..., must now be edge dislocations.

For three-dimensional situations of the kind shown in figure 1 it may be possible to use a Volterra dislocation to approximate to a Somigliana dislocation. For example (3.1) will give the correct elastic field at *remote* points if $\boldsymbol{b}$ is taken to be a suitable average of $\boldsymbol{b}(\boldsymbol{r})$ over the surface S of the Somigliana dislocation. If something better is needed a Somigliana dislocation can be synthesized from Volterra dislocations. The method illustrated in figure 3*b* can be extended to situations which are not two-dimensional (Leibfried 1954; Eshelby 1963), but it is not very advantageous. It is better to regard the discontinuity surface of the Somigliana dislocation as divided into a mosaic of small areas across each of which b is sensibly constant (cf. the remark preceding (2.4)). Then each elementary area is equivalent to a small dislocation loop to which the appropriate formula may be applied.

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